The other derivative may be developed as follows:

$$\frac{\partial}{\partial t}\left[\frac{\partial R(r,\theta)}{\partial \xi}\right]_0 = \overline{v\frac{\partial^2 v}{\partial x \partial t}}$$

$$= \overline{\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}} + \overline{\frac{\partial}{\partial t}\left(v\frac{\partial v}{\partial x}\right)} \qquad \text{(A-9)}$$

For stationary turbulence the Eulerian space derivative is not a function of time, so

$$\left.\frac{\partial^2 R(r,\theta)}{\partial \xi \partial t}\right|_0 = -\overline{\frac{\partial v}{\partial t}\frac{\partial v}{\partial x}} \qquad \text{(A-10)}$$

Combining Equations (A-5), (A-7), (A-8), and (A-10) and noting that Equation (A-1) may be used to eliminate the trigonometric functions one obtains

$$R_v(r,\theta) = 1 - \frac{\tau^2}{2\overline{v^2}}\left[\overline{V^2\left(\frac{\partial v}{\partial x}\right)^2} + 2V\overline{\frac{\partial v}{\partial x}\frac{\partial v}{\partial t}} + \overline{\left(\frac{\partial v}{\partial t}\right)^2}\right] + \ldots \qquad \text{(A-11)}$$

For the special case of θ, where $V = U$, Equation (A-11) becomes

$$R_v(\xi,\tau)\Big|_{\xi=U\tau} = 1 - \frac{\tau^2}{2\overline{v^2}}\left[U^2\overline{\left(\frac{\partial v}{\partial x}\right)^2} + 2U\overline{\frac{\partial v}{\partial x}\frac{\partial v}{\partial t}} + \overline{\left(\frac{\partial v}{\partial t}\right)^2}\right] + \ldots \qquad \text{(A-12)}$$

When one recalls Burger's approximation of the Lagrangian derivative

$$\frac{dv}{dt} \cong \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} \qquad \text{(A-13)}$$

or

$$\overline{\left(\frac{dv}{dt}\right)^2} \cong U^2\overline{\left(\frac{\partial v}{\partial x}\right)^2} + 2U\overline{\frac{\partial v}{\partial x}\frac{\partial v}{\partial t}} + \overline{\left(\frac{\partial v}{\partial t}\right)^2} \qquad \text{(A-13a)}$$

Substituting Equation (A-13a) into (A-12) and assuming the approximate equivalence of single-particle time averages to two-particle space averages one gets

$$R_v(\xi,\tau)\Big|_{\xi=U\tau} \cong 1 - \frac{\tau^2}{2\overline{v^2}}\overline{\left(\frac{dv}{dt}\right)^2} + \ldots \qquad \text{(A-14)}$$

Or letting $\overline{v^2} \cong \overline{v^2_{Lo}}$ one gets the desired relation:

$$R_v(\xi,\tau)\Big|_{\xi=U\tau} \cong \mathcal{R}(\tau) \qquad \text{(A-15)}$$

Equation (A-15) verifies the remark that to the same approximation as Equation (A-13) the general Eulerian correlation along $\xi = U\tau$ will approach the true Lagrangian correlation coefficient in a homogeneous, isotropic, and stationary turbulence.

Physically $R_v(\xi,\tau)\Big|_{\xi=U\tau}$ corresponds to the Eulerian autocorrelation as measured by a fixed probe in a hypothetical box of turbulence which is homogeneous, isotropic, and stationary. Bass (*40*) has given a theoretical development for Eulerian space-time correlations which is comparable to that of Kármán and Howarth (*5*) for the more familiar case of Eulerian space correlations.

A Modification of the Momentum Transport Hypothesis

WILLIAM N. GILL and MARVIN SCHER

Syracuse University, Syracuse, New York

A continuous velocity distribution is derived which is based on an arbitrary modification of Prandtl's mixing length expression. The resulting velocity distribution agrees well with experiments for transition and fully developed turbulent flow throughout the entire cross section of the conduit. Furthermore the mixing length expression applies to parallel flow in smooth circular tubes and between infinite parallel plates with the same set of constants.

Prandtl's famous momentum transport hypothesis has been successful in predicting velocity distributions in the turbulent core of bounded or conduit flow. Usually velocity distribution expressions derived on the basis of this hypothesis neglect the shear stress variation across the conduit, the molecular viscosity term, and more important the viscous dampening effect of the wall on eddy properties near the wall. Van Driest (*15*) considered viscous dampening to be an exponential function of the distance from the wall and showed how this dampening is influenced by surface roughness. However the extent of viscous dampening has been clearly demonstrated by the experimental work of Sage and coworkers (*1, 8, 9*) to be a function of the Reynolds number and has not been successfully explained except on an empirical basis by Rothfus and Monrad (*11*). Consequently a derivation of the generalized velocity distribution expression is needed which accounts for the previously mentioned effects within the framework of Prandtl's hypothesis. The following treatment has several advantages over existing velocity distribution expressions which are:

1. A single continuous function represents the velocity distribution from the wall to the center of the channel for both flat plates and circular tubes.
2. A zero velocity gradient at the center of the channel is obtained.
3. The Reynolds number effect on the velocity distribution is accounted for and will be seen to be most pronounced in the transition flow region.
4. The velocity distribution reduces to the Hagen-Poiseuille equation at $N_{Re} = 1{,}800$ for circular tubes and at $N_{Re} = 4{,}800$ for flat plates.

It has been shown that eddy properties very near a solid boundary may be more precisely determined from heat and mass transfer measurements at high Schmidt or Prandtl numbers than by velocity distribution studies (*3, 5, 10*). However this discussion is intended only to describe these approximately in the immediate vicinity of the wall and at the same time resolve the apparent discrepancies between turbulent transition flow in tubes and between parallel plates which have been observed as a result of careful measurements. Velocity distribution explorations by Deissler (*2*) and Laufer (*4*) for turbulent flow in tubes and between flat plates indicate a single valued relation between u^+ and y^+ over

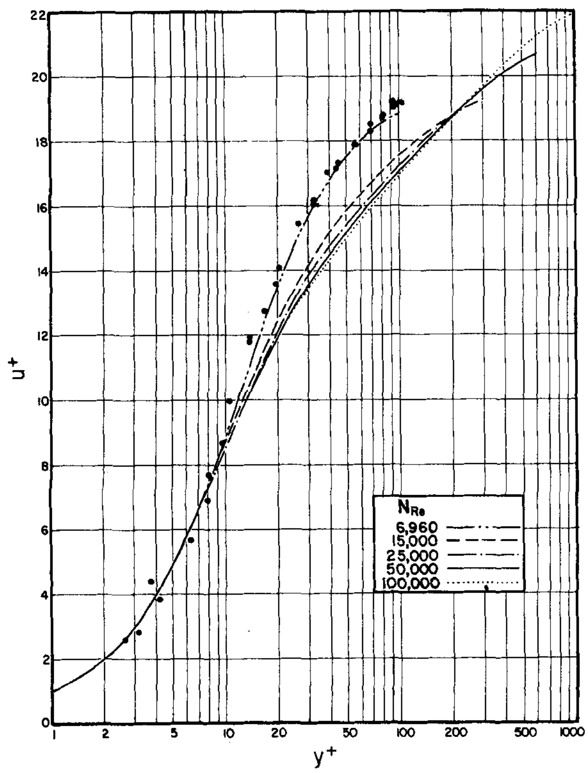

Fig. 1. Velocity distribution between smooth infinite parallel plates with N_{Re} as parameter.

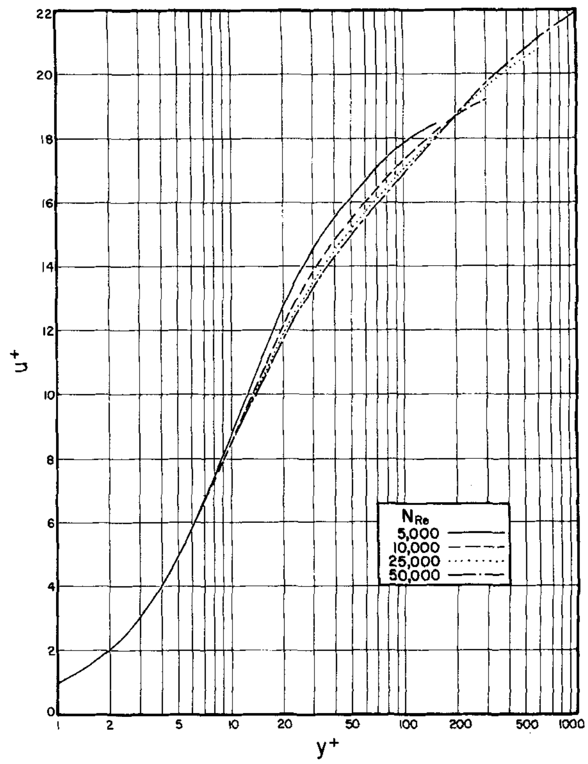

Fig. 2. Velocity distribution in smooth circular tubes with N_{Re} as parameter.

the entire Reynolds number range investigated. In contrast the results obtained by Sage et al. (*1, 8, 9*), for parallel plates show a definite Reynolds number effect in the transition region up to Reynolds number of approximately 30,000. The effect observed by Sage can be explained by considering that the wall inhibits or dampens turbulence and that the depth of penetration of this dampening effect into the stream depends on the Reynolds number. At low N_{Re} the dampening can be effective throughout the entire channel, but as N_{Re} increases the region influenced is confined to distances from the wall on the order of $y^+ = 60$.

MODIFICATION OF THE MIXING LENGTH

The time average turbulent shear stress $\rho\,\overline{u'v'}$ combined with the molecular shear stress gives, after introduction of the Prandtl mixing length

$$\tau = \mu \frac{du}{dy} + \rho \left(l \frac{du}{dy} \right)^2 \quad (1)$$

Prandtl reasoned that $l = ky$; however this assumption leads to irreconcilable differences between experiment and theory in the vicinity of the wall, and it has been customary to apply the results only in the turbulent core where $y^+ > 26$. However the transition velocity profiles of Sage et al. indicate that the results agree with experiment only at Reynolds numbers above approximately 30,000 even in the central portion of the conduit. It is proposed to account for these deviations in terms of a wall dampening effect on the mixing length which is a function of N_{Re}. Therefore the following mixing length expression is assumed:

$$l = ky \left(1 - e^{-\phi \frac{y}{y_m}} \right) \quad (2)$$

where

$$\phi = \frac{N^*_{Re} - a}{b}$$

and

$$N^*_{Re} = \frac{y_m u^* \rho}{\mu} = y^+_m$$

Correspondingly the eddy diffusivity expression in dimensionless form is given by

$$\epsilon/\nu = k^2 y^{+2} \left(1 - e^{-\phi \frac{y^{+2}}{y^+_m}} \right)^2 \frac{du^+}{dy^+} \quad (3)$$

The experimental data of Senecal (*14*) indicate that the Hagen-Poiseuille equation describes flow in tubes well up to Reynolds number of approximately 1,800. Therefore it will be assumed that eddy effects are completely damped out at this value which requires that $(1 - e^{-\phi \frac{y^+}{y^+_m}})$ vanish for all y^+ values. This is conveniently accomplished by setting the constant a equal to 60 which for tubes corresponds to y^+_m for N_{Re} equal to 1,800. The constant b must be determined from experimental data and was found to be equal to 22 to obtain good agreement with the data of Nikuradse (*7*), and Deissler (*2*), in the higher N_{Re} range. This constant may be considered as a measure of the depth of penetration of viscous dampening into the stream due to the wall. In accordance with Deissler (*2*) k was taken to be 0.36.

DERIVATION OF THE GENERALIZED VELOCITY DISTRIBUTION

When one divides Equation (1) by the shear stress at the wall and employs the linear variation of shear stress with position, the result combined with Equation (3) yields

$$k^2 y^{+2} \left(1 - e^{-\phi \frac{y^{+2}}{y^+_m}} \right)^2 \left(\frac{du^+}{dy^+} \right)^2 + \frac{du^+}{dy^+} = 1 - \frac{y^+}{y^+_m} \quad (4)$$

For convenience let

$$c = k^2 y^{+2} \left(1 - e^{-\phi \frac{y^{+2}}{y^+_m}} \right)^2$$

$$d = 1 - \frac{y^+}{y^+_m}$$

and it is seen that Equation (4) is obviously a quadratic equation in du^+/dy^+ and may be solved to give

$$\frac{du^+}{dy^+} = \frac{-1 + \sqrt{1 + 4cd}}{2c} \quad (5)$$

where the positive sign for the radical was chosen so that du^+/dy^+ is always positive. Integrating one obtains

$$u^+ = \int_0^{y^+} \frac{-1 + \sqrt{1 + 4cd}}{2c} dy^+ \quad (6)$$

RESULTS

The results of the numerical integration of Equation (6) are given in Figures 1 and 2 for infinite parallel plates, and smooth circular tubes, respectively. Figure 1 indicates rather remarkable agreement with the experimental results of Sage et al. (*1*, *8*, *9*). Some experimental values at N_{Re} = 6,960 have been plotted for comparison, and Sage's data at higher N_{Re} converge in the same manner as the analytical results. It is seen that for low N_{Re} the distributions differ markedly from the logarithmic expressions of Deissler (*2*) and Nikuradse (*7*), but the distributions converge to values between their expressions for N_{Re} above 25,000. Also near the center of the channel the velocity profile flattens owing to the linear shear stress distribution. As seen in Figure 2 the N_{Re} effect is much less pronounced for tubes, and the velocity distributions converge more rapidly. The differences between the results for the two configurations arises from ϕ, since y^+_m may be obtained for tubes as

$$y^+_m = \frac{N_{Re}}{2} \sqrt{f/2}$$

and for plates by

$$y^+_m = \frac{N_{Re}}{4} \sqrt{f/2}$$

The asymptotic behavior of Equation (6) deserves some consideration, and for this purpose Equation (6) can be rearranged to the more convenient form

$$u^+ = \int_0^{y^+} \frac{d}{1/2 + \frac{\sqrt{1 + 4cd}}{2}} dy^+ \quad (7)$$

For very small values of y^+, $c \approx 0$, and $d \approx 1$. Therefore Equation (7) reduces to

$$u^+ = y^+$$

At reasonably high N_{Re} and y^+ on the order of 60 or more the integrand of Equation (7) reduces to $\sqrt{d/c}$ which leads to the velocity deficiency relation

$$u^+_m - u^+ = \frac{1}{k} \ln \frac{1 + \sqrt{1 - y^+/y^+_m}}{1 - \sqrt{1 - y^+/y^+_m}} - \frac{2}{k} \sqrt{1 - y^+/y^+_m}$$

If the variation in shear stress is neglected ($d = 1$), the logarithmic distribution results in

$$u^+ = \frac{1}{k} \ln y^+ + \text{constant}$$

These asymptotic results are, of course, those that would be expected.

Recently Lynn (*6*) commented on an article by Rothfus, Archer, and Sikchi (*12*) regarding eddy diffusivity distributions and indicated that ϵ/ν near the center of a channel was not zero. Combining Equations (3) and (5) one can see that the eddy diffusivity is given by the well-defined expression

$$\epsilon/\nu = \frac{-1 + \sqrt{1 + 4cd}}{2} \quad (8)$$

Therefore on the basis of Equation (8), for fully developed channel flow where the shear stress is linear in the radius, ϵ/ν is predicted to be zero at the center line. This discrepancy between Prandtl's theory and experiment, where velocity gradients become zero, was recognized by Prandtl as well, and in 1942 he proposed an alternate expression for the eddy diffusivity (*13*) which however complicates the calculation considerably and will not be considered here.

CONCLUSIONS

If the mixing length is assumed to be a function of both the Reynolds number and the distance from the wall, a continuous velocity distribution may be derived which is applicable to turbulent flow in smooth circular tubes and between infinite parallel plates. Furthermore the resulting expression converges to the Hagen-Poiseuille equation at appropriate Reynolds numbers and therefore represents transition flow for both configurations. In contrast to previously proposed universal velocity distributions it is not necessary to subdivide the flow in separate regions, and therefore this approach presents a physically more realistic model.

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NOTATION

a	=	dimensionless constant equal to 60
b	=	dimensionless constant equal to 22
d_e	=	equivalent diameter (L)
f	=	fanning friction factor
k	=	dimensionless constant equal to 0.36
l	=	Prandtl mixing length (L)
U	=	bulk mean velocity $(L\theta^{-1})$
u,v	=	time average x and y components of velocity $(L\theta^{-1})$
u',v'	=	fluctuating x and y components of velocity $(L\theta^{-1})$
u^+	=	dimensionless velocity parameter
u^*	=	friction velocity $(u/\sqrt{\tau_o/\rho})$ $(L\theta^{-1})$
y	=	distance from the wall (L)
y^+	=	dimensionless distance parameter $(u^* y\rho/\mu)$

Dimensionless Groups

N_{Re}	=	Reynolds number $(Ud_e\rho/\mu)$

Subscripts

m	=	maximum
t	=	turbulent
o	=	wall value

Greek Letters

ϵ	=	eddy viscosity $(ML^{-1}\theta^{-1})$
μ	=	molecular viscosity $(ML^{-1}\theta^{-1})$
ρ	=	density (ML^{-3})
ν	=	kinematic viscosity $(L^2\theta^{-1})$
τ	=	shear stress $(ML^{-1}\theta^{-2})$

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